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Hall Effect on Unsteady Hartmann Flow with Heat Transfer Under Exponential Decaying Pressure Gradient

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The unsteady Hartmann flow of an electrically conducting, viscous, incompressible fluid bounded by two parallel non-conducting porous plates is studied with heat transfer taking the Hall effect into consideration. An external uniform magnetic field and a uniform suction and injection are applied perpendicular to the plates while the fluid motion is subjected to an exponential decaying pressure gradient. The two plates are kept at different but constant temperatures while the Joule and viscous dissipations are included in the energy equation. The effect of the ion slip and the uniform suction and injection on both the velocity and temperature distributions is examined.

1. Introduction

The magnetohydrodynamic flow between two parallel plates, known as Hartmann flow, is a classical problem that has many applications in magnetohydrodynamic (MHD) power generators, MHD pumps, accelerators, aerodynamic heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil and fluid droplets and sprays. Hartmann and Lazarus (1937) studied the influence of a transverse uniform magnetic field on the flow of a conducting fluid between two infinite parallel, stationary, and insulated plates. Then, a lot of research work concerning the Hartmann flow has been obtained under different physical effects (Tao, 1960 ; Alpher, 1961 ; Sutton and Sherman, 1965 ; Cramer and Pai, 1973 ; Nigam and Singh, 1960 ; Tani, 1962 ; Soundalgekar et al., 1979 ; Soundalgekar and Uplekar, 1986 ; Abo-El-Dahab, 1993). In most cases the Hall and ion slip terms were ignored in applying Ohm's law as they have no marked effect for small and moderate values of the magnetic field. However, the current trend for the application of magnetohydrodynamics is towards a strong magnetic field, so that the influence of electromagnetic force is noticeable (Cramer and Pai, 1973). Under these conditions, the Hall current and ion slip are important and they have a marked effect on the magnitude and direction of the current density and consequently on the magnetic force term. Tani (1962) studied the Hall effect on the steady motion of electrically conducting and viscous fluids in channels. Soudalgekar et al. (1979 ; Soundalgekar and Uplekar, 1986) studied the effect of the Hall currents on the steady MHD Couette flow with heat transfer. The temperatures of the two plates were assumed either to be constant (Soudalgekar et al., 1979) or to vary linearly along the plates in the direction of the flow (Soundalgekar and Uplekar, 1986). Abo-El-Dahab (1993) studied the effect of Hall current on the steady Hartmann flow subjected to a uniform

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suction and injection at the bounding plates. Later, Attia (1998) extended the problem to the unsteady state with heat transfer, with constant pressure gradient applied.

In the present study, the unsteady flow and heat transfer of an incompressible, viscous, electrically conducting fluid between two infinite non-conducting horizontal porous plates are studied with the consideration of the Hall current. The fluid is acted upon by an exponential decaying pressure gradient, a uniform suction and injection and a uniform magnetic field perpendicular to the plates. The equations of motion are solved analytically using the Laplace transform method while the energy equation is solved numerically taking the Joule and the viscous dissipations into consideration. The effect of the magnetic field, the Hall current, the ion slip, and the suction and injection on both the velocity and temperature distributions is studied.

2. Description of the Problem

The two non-conducting plates are located at the $y=\pm h$ planes and extend from $x=-\infty$ to ∞ and $z=-\infty$ to ∞ as shown in Fig. 1. The lower and upper plates are kept at the two constant temperatures T_1 and T_2, respectively, where $T_2>T_1$. The fluid flows between the two plates under the influence of an exponential decaying pressure gradient dP/dx in the x-direction, and a uniform suction from above and injection from below which are applied at $t=0$. The whole system is subjected to a uniform magnetic field B_o in the positive y-direction. This is the total magnetic field acting on the fluid since the induced magnetic field is neglected (Sutton and Sherman, 1965; Cramer and Pai, 1973). From the geometry of the problem, it is evident that all quantities apart from the pressure gradient dP/dx do no not depend upon x or z. The existence of the Hall term gives rise to a z-component of the velocity. This configuration is a good approximation of some practical situations such as heat exchangers, flow meters, and pipes that connect system components. The cooling of these devices can be achieved by utilizing a porous surface through

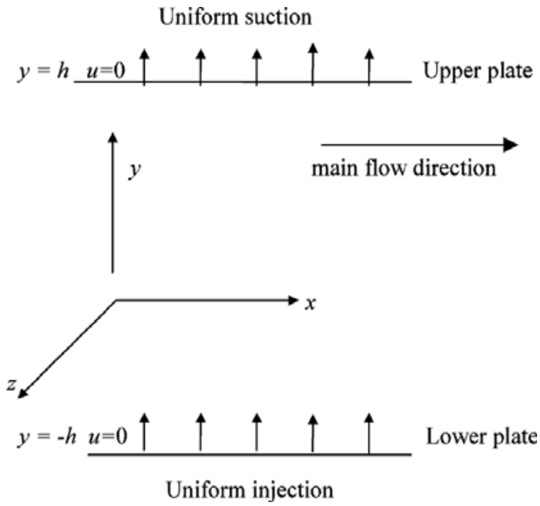


Fig. 1 Schematic diagram of the problem

which a coolant, either a liquid or gas, is forced. Therefore, the results obtained here are important for the design of the wall and the cooling arrangements of these devices. Thus, the velocity vector of the fluid is

$$\vec{v}(y,\ t)=u(y,\ t)\vec{i}+v_o\vec{j}+w(y,\ t),\ \vec{k}$$

with the initial and boundary conditions $u=w=0$ at $t\le 0$, and $u=w=0$ at for $t>0$. The temperature $T(y,\ t)$ at any point in the fluid satisfies both the initial and boundary conditions $T=T_1$ at $t\le 0$, $T=T_2$ at $y=+h$, and $T=T_1$ at $y=-h$ for $t>0$. The fluid flow is governed by the momentum equation

$$\rho\frac{D\vec{v}}{Dt}=\mu\nabla_2\vec{v}-\vec{\nabla}P+\vec{J}\wedge\vec{B}_o \tag{1}$$

where ρ and μ are, respectively, the density and the coefficient of viscosity of the fluid. If the Hall term is retained, the current density $\vec{J}$ is given by

$$\vec{J}\sigma\{\vec{v}\wedge\vec{B}_o-\beta(\vec{J}\wedge\vec{B}_o)\}$$

where σ is the electric conductivity of the fluid, and β is the Hall factor (Sutton and Sherman, 1965). This equation may be solved in **J** to yield

$$\vec{J}\wedge\vec{B}_o=-\frac{\sigma B_o^2}{1+m^2}\{(u+mw)\vec{i}+(w-mu)\vec{k}\} \tag{2}$$

where $m=\sigma\beta B_o$, is the Hall parameter (Sutton and Sherman, 1965). Thus, in terms of Eq. (2), the two components of Eq. (1) read

$$\rho\frac{\partial u}{\partial t}+\rho v_o\frac{\partial u}{\partial y}=-\frac{dP}{dx}+\mu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_o^2}{1+m^2}(u+mw) \quad (3)$$

$$\rho\frac{\partial w}{\partial t}+\rho v_o\frac{\partial w}{\partial y}=\mu\frac{\partial^2 w}{\partial y^2}-\frac{\sigma B_o^2}{1+m^2}(w-mu) \quad (4)$$

To find the temperature distribution inside the fluid we use the energy equation (Schlichting, 1968)

$$\rho c\frac{\partial T}{\partial t}+\rho c v_o\frac{\partial T}{\partial y}=k\frac{\partial^2 T}{\partial y^2}+\mu\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] +\frac{\sigma B_o^2}{1+m^2}(u^2+w^2) \quad (5)$$

where c and k are, respectively, the specific heat capacity and the thermal conductivity of the fluid. The second and third terms on the right-hand side represent the viscous and Joule dissipations, respectively.

The problem is simplified by writing the equations in the non-dimensional form. The characteristic length is taken to be h, and the characteristic time is $\rho h^2/\mu^2$ while the characteristic velocity is $\mu/\rho h$. We define the following non-dimensional quantities

$$\hat{x}=\frac{x}{h},\ \hat{y}=\frac{y}{h},\ \hat{z}=\frac{z}{h},\ \hat{u}=\frac{\rho h u}{\mu},\ \hat{w}=\frac{\rho h w}{\mu},$$
$$\hat{P}=\frac{P\rho h^2}{\mu^2},\ t=\frac{t\mu}{\rho h^2}$$

$S=\rho v_o h/\mu$ is the suction parameter,
$\mathrm{Pr}=\mu c/k$ is the Prandtl number,
$Ec=\mu^2/\rho^2 c h^2(T_2-T_1)$ is the Eckert number,
$Ha^2=\sigma B_o^2 h^2/\mu$ where Ha is the Hartmann number,

In terms of the above non-dimensional variables and parameters, the basic Eqs. (3)-(5) are written as (the "hats" will be dropped for convenience)

$$\frac{\partial u}{\partial t}+S\frac{\partial u}{\partial y}=-\frac{dP}{dx}+\frac{\partial^2 u}{\partial y^2}-\frac{Ha^2}{1+m^2}(u+mw) \quad (6)$$

$$\frac{\partial w}{\partial t}+S\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}-\frac{Ha^2}{1+m^2}(w-mu) \quad (7)$$

$$\frac{\partial T}{\partial t}+S\frac{\partial T}{\partial y}=\frac{1}{\mathrm{Pr}}\frac{\partial^2 T}{\partial y^2}+Ec\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] +\frac{EcHa^2}{1+m^2}(u^2+w^2) \quad (8)$$

The initial and boundary conditions for the velocity become

$$u=w=0,\ t\le 0,\ u=w=0,\ y=\pm 1,\ t>0 \quad (9)$$

and the initial and boundary conditions for the temperature are given by

$$t\le 0:\ T=0,\ t>0:\ T=1,\ y=+1,\ T=0,\ y=-1 \quad (10)$$

3. Analytical Solution of the Equations of Motion

Equations (6) and (7) are the two equations of motion which, if solved, give the two components of the velocity field as functions of space and time. Multiplying Eq. (7) by i and adding to Eq. (6) we obtain

$$\frac{\partial^2 V}{\partial y^2}-S\frac{\partial V}{\partial y}-\frac{Ha^2(1-im)}{1+m^2}V-\frac{\partial V}{\partial t}=\frac{dP}{dx} \quad (11)$$

with the initial and boundary conditions

$$V=0,\ t\le 0,\ V=0,\ y=\pm 1,\ t>0 \quad (12)$$

where $V=u+iw$. Equations (11) and (12) can be solved using the method of Laplace Transform (LT) (Spiegel, 1986) to obtain V as functions of y and t. The real part of V or V_p represents the x-component of the velocity while the imaginary part represents the z-component. Taking LT of Eqs. (11) and (12) we have

$$\frac{d^2\bar{V}(y,s)}{dy^2}-S\frac{d\bar{V}(y,s)}{dy} -K(s)\bar{V}(y,s)=-F(s) \quad (13)$$

where $\bar{V}(y,s)=L(V(y,t))$, $-F(s)$ is the LT of the pressure gradient, $K(s)=A+s$, and $A=Ha^2(1-im)/(1+m^2)$. The solution of Eq. (13) with y as an independent variable is given as

$$\bar{V}(y,s)=\frac{F(s)}{K}\left(1+\exp(Sy/2)\left[\frac{\sinh(S/2)\sinh(qy)}{\sinh(q)} -\frac{\cosh(S/2)\cosh(qy)}{\cosh(q)}\right]\right)$$

where $q^2=S^2/4+K$. Using the complex inversion formula and the residue theorem (Spiegel, 1986), the inverse transform of $\bar{V}(y, s)$, is determined as

$$V(y, t)=C\sum_{n=1}^{\infty}\Big(\frac{PI_1}{PN_1+\alpha}(\exp(PN_1xt)-\exp(-\alpha t)) + \frac{PI_2}{PN_2+\alpha}(\exp(PN_2xt)-\exp(-\alpha t)) + \frac{PI_3}{PN_3+\alpha}(\exp(PN_3xt)-\exp(-\alpha t)) + \frac{PI_4}{PN_4+\alpha}(\exp(PN_4xt)-\exp(-\alpha t))\Big) \quad (14)$$

where

$$-\frac{dP}{dx}=C\exp(-\alpha t)$$

$$PN_1=PN_2=NN_1/2$$

$$PN_3=PN_4=NN_2/2$$

$$PI_1=\frac{NN_3}{A+PN_1}$$

$$PI_2=\frac{NN_3}{A+PN_2}$$

$$PI_3=\frac{NN_4}{A+PN_3}$$

$$PI_4=\frac{NN_4}{A+PN_4}$$

$$NN_1=-\pi^2(n-1)^2-S^2/4$$

$$NN_2=-\pi^2(n-0.5)^2-S^2/4$$

$$NN_3=2\pi(-1)^n(n-1)\exp(Sy/2)\sinh(S/2)\sin(\pi(n-1)y)$$

$$NN_4=2\pi(-1)^{n+1}(n-0.5)\exp(Sy/2)\cos(S/2)\cos(\pi(n-0.5)y)$$

The expression for the complex velocity V is to be evaluated for different values of the parameters Ha, m and S. The velocity components u and w are, respectively, the real and imaginary parts of V.

4. Numerical Solution of the Energy Equation

The exact solution of the equations of motion, given by Eq. (13), determines the velocity field for different values of the parameters Ha, m and S. The values of the velocity components, when substituted in the right-hand side of the inhomogeneous energy equation (8), make it too difficult to solve analytically. The energy equation is to be solved numerically with the initial and boundary conditions given by Eq. (10) using finite differences (Ames, 1977). The Crank-Nicolson implicit method is applied. The finite difference equations are written at the mid-point of the computational cell and the different terms are replaced by their second-order central difference approximations in the y-direction. The diffusion term is replaced by the average of the central differences at two successive time levels. The viscous and Joule dissipation terms are evaluated using the velocity components and their derivatives in the y-direction which are obtained from the exact solution. Finally, the block tri-diagonal system is solved using Thomas' algorithm.

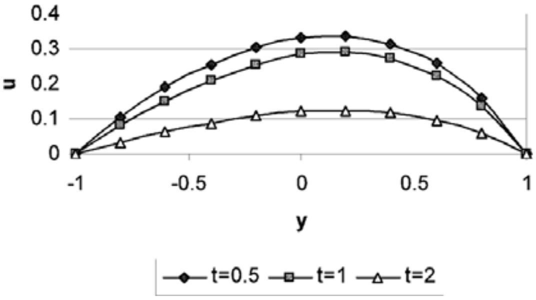
5. Results and Discussion

Figure 2 shows the profiles of the velocity components u and w and temperature T for various values of time t. The figure is plotted for $Ha=1$, $m=3$ and $S=1$. As shown in Fig. 2(a) and 2(b), the profiles of u and w are asymmetric about the plane $y=0$ because of the suction. It is observed that the velocity components and temperature reaches the steady state monotonically with time.

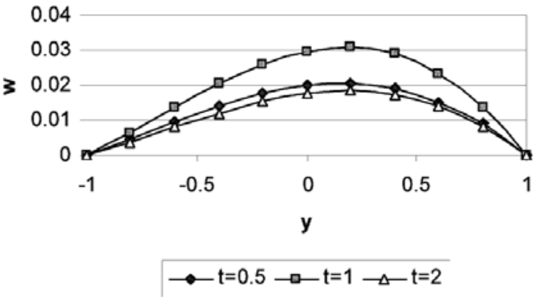
Figure 3 shows the time evolution of u and w at the centre of the channel $y=0$ for various values of the Hall parameter m. In this figure, $Ha=1$ and $S=0$. It is clear from Fig. 3(a) that increasing the parameter m increases u. This is because the effective conductivity $(\sigma/(1+m^2))$ decreases with increasing m which reduces the magnetic damping force on u. In Fig. 3(b), the velocity component w increases with increasing the parameter m slightly ($m=0$ to 1), since increasing m increases the driving force term $(mHa^2u/(1+m^2))$ in Eq. (7) which pumps the flow in the z-direction. However, increasing m more decreases the effective conductivity that results in a reduced driving force and then, decreases w.

Table 1 Time variation of the temperature at $y=0$ for various values m ($S=0$, $Ha=1$)

T	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1$	$t=1.2$	$t=1.4$	$t=1.6$	$t=1.8$	$t=2$
$m=0$	0.1159	0.2675	0.3617	0.4187	0.4529	0.4730	0.4848	0.4916	0.4955	0.4976
$m=1$	0.1158	0.2669	0.3610	0.4182	0.4526	0.4729	0.4849	0.4917	0.4956	0.4978
$m=3$	0.1156	0.2664	0.3602	0.4175	0.4521	0.4727	0.4848	0.4918	0.4957	0.4979



(a)



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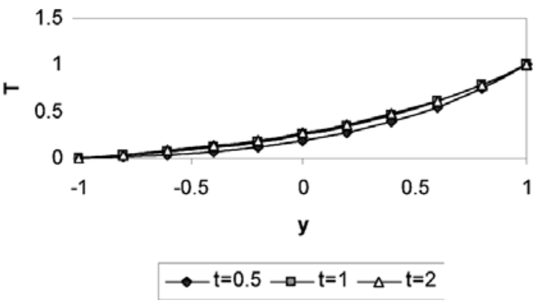
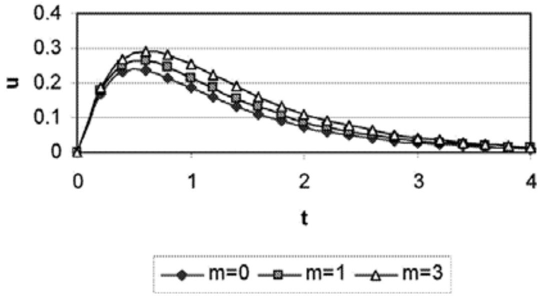


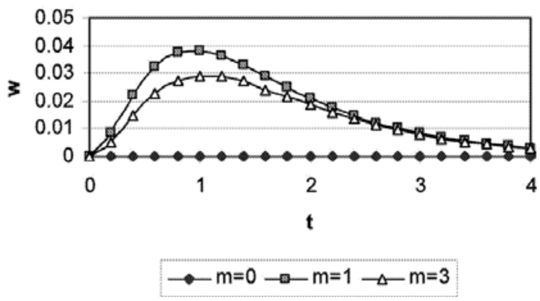
(c)

Fig. 2 Time development of the profile of: (a) u; (b) w; and (c) T ($Ha=1$, $m=3$ and $S=1$)

Table 1 presents the time evolution of the temperature T at the centre of the channel $y=0$ for various values of the Hall parameter m. It is clear from Table 1 that, for small t, increasing m decreases T as increasing m reduces the effect of the Joule dissipation. However, for large t,



(a)



(b)

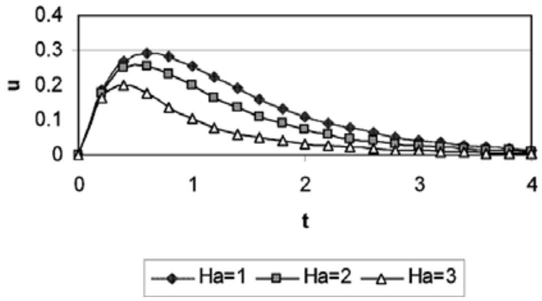
Fig. 3 Effect of m on the time variation of: (a) u at $y=0$; (b) w at $y=0$ ($Ha=1$ and $S=0$)

increasing m increases T since increasing m increases the main flow velocity u which develops with time and therefore increases the dissipations.

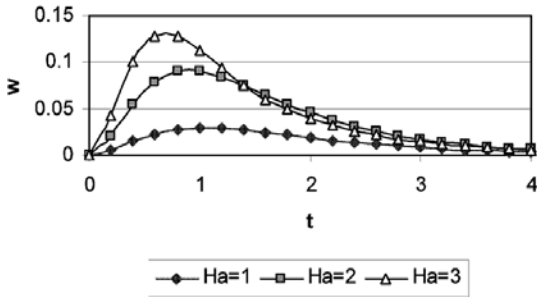
Figure 4 shows the time evolution of u and w at the centre of the channel $y=0$ for various values of the Hartmann number Ha. In this figure, $m=3$ and $S=0$. Figure 4(a) indicates that increasing Ha decreases u as a result of increasing the damping force on u. Figure 4(b) shows that increasing Ha increases w since it increases the damping force on w. However, increasing Ha more increases w at small t but decreases it at large t. This can be attributed to the fact that large Ha decreases the main velocity u, which increases with time, and reduces the driving force

Table 2 Time variation of the temperature T at $y=0$ for various values Ha ($S=0$, $m=3$)

T	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1$	$t=1.2$	$t=1.4$	$t=1.6$	$t=1.8$	$T=2$
$Ha=1$	0.1156	0.2664	0.3602	0.4175	0.4521	0.4727	0.4848	0.4918	0.4957	0.4979
$Ha=2$	0.1157	0.2668	0.3608	0.4179	0.4523	0.4726	0.4845	0.4914	0.4953	0.4975
$Ha=3$	0.1159	0.2673	0.3612	0.4178	0.4516	0.4716	0.4834	0.4903	0.4944	0.4968

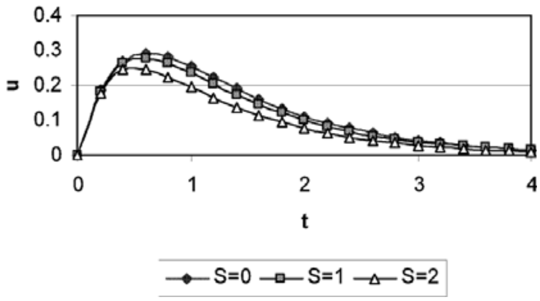


(a)

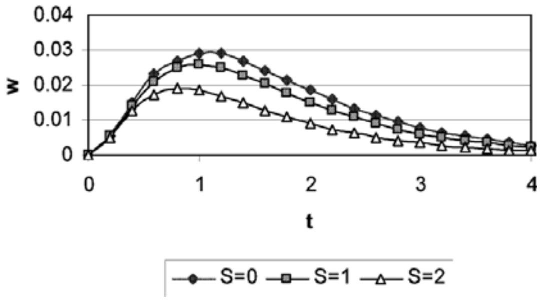


(b)

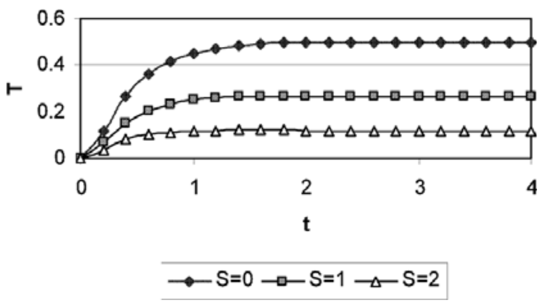
Fig. 4 Effect of Ha on the time variation of: (a) u at $y=0$; (b) w at $y=0$ ($m=3$ and $S=0$)



(a)



(b)



(c)

Fig. 5 Effect of m on the time variation of: (a) u at $y=0$; (b) w at $y=0$; and (c) T at $y=0$ ($Ha=1$ and $m=3$)

on w which results in decreasing w at large t. Table 2 presents the time evolution of T at the centre of the channel $y=0$ for various values of the Hartmann number Ha. It is clear that for small t, increasing Ha increases T due to increasing the Joule dissipation. But, for large t, increasing Ha decreases T as a result of decreasing the velocities u and w and consequently decreases the viscous and Joule dissipations.

Figure 5 presents the time evolution of u, w and T at the centre of the channel $y=0$ for various values of the suction parameter S. In this figure $Ha=1$ and $m=3$. Figures 5(a) and 5(b) show that increasing the suction decreases both u and w due to the convection of the fluid from regions in the lower half to the centre which has higher fluid speed. Figure 5(c) shows that increasing S decreases the temperature at the centre of the channel. This is due to the influence f convection in pumping the fluid from the cold lower half towards the centre of the channel.

6. Conclusions

The unsteady Hartmann flow of a conducting fluid under the influence of an applied uniform magnetic field has been studied considering the Hall effect in the presence of uniform suction and injection and an exponential decaying pressure gradient. An analytical solution was obtained for the momentum equations while the energy equation including the viscous and Joule dissipations was solved numerically. Introducing the Hall term gives rise to a velocity component w in the z-direction and it affected the main velocity u in the x-direction. The effect of the magnetic field, the Hall parameter and the suction and injection velocity on the velocity and temperature distributions has been investigated. As time develops, increasing the Hall parameter m increases the velocity component u and increases the velocity component w for small m and decreases it for large m. Also, it is found that the effect of large Ha on w depends on time. It is found also, that the influence of both parameters Ha and m on the temperature T depends on time.

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